Student Number ....

Barker College

2009
TRIAL
HIGHER SCHOOL CERTIFICATE

Mathematics Extension 1

Staff Involved:

- **LJP***
- **PJR***
- **MRB**
- **GDH**
- **WMD**
- **RMH**
- **BTP**

PM THURSDAY 20 AUGUST

75 copies

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Make sure your Barker Student Number is on ALL pages of your solutions
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- ALL necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value
- Marks may be deducted for careless or poorly arranged working

Total marks – 84
Attempt Questions 1–7
ALL questions are of equal value

Answer each question on a SEPARATE sheet of paper

Marks

Question 1 (12 marks) **[START A NEW PAGE]**

(a) Find the coordinates of the point P which divides the interval joining A (-2, 3) and B (3, -4) externally in the ratio 3 : 2 **2**

(b) Evaluate $\lim_{x \to 0} \left(\frac{\sin 5x}{2x} \right)$ **2**

(c) Solve $\frac{3}{2x-4} > -2$ **3**

(d) Evaluate $\int_{-1}^{1} \frac{-1}{\sqrt{2-x^2}}\, dx$ **2**

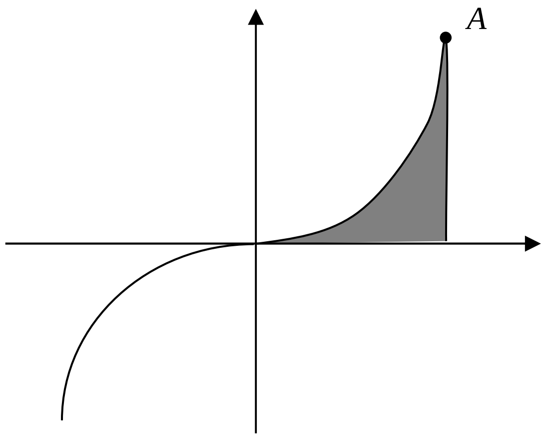
(e) Evaluate $\int_{0}^{\frac{\pi}{12}} 2\sin^2 4x \, dx$ **3**

Marks

Question 2 (12 marks) **[START A NEW PAGE]**

(a) Find $\int 6x^3 \sqrt{\left(3x^4 - 3\right)^3}\, dx$ using the substitution $u = 3x^4 - 3$ **2**

(b) The graph of $y = 2\sin^{-1}\left(\frac{x}{3}\right)$ is shown below



(i) Write down the coordinates of Point A **2**

(ii) Differentiate $y = 2x\sin^{-1}\left(\frac{x}{3}\right) + 2\sqrt{9 - x^2}$ **2**

(iii) Hence, or otherwise, find the shaded area **2**

(c) (i) Express $\sqrt{3}\sin t + \cos t$ in the form $R\sin\left(t + \alpha\right)$ **2**

where α is in radians

(ii) Hence, or otherwise, find the solutions of the equation **2**

$\sqrt{3}\sin t + \cos t = \sqrt{3}$ for $0 \leq t \leq 2\pi$

Marks

Question 3 (12 marks) **[START A NEW PAGE]**

(a) Consider the function $y = (x+2)^2 + 1$

(i) Write down the entirely negative domain for which the inverse function exists **1**

(ii) State the domain of this inverse function **1**

(b) Find the constant term in the expansion $\left(x^4 + \dfrac{3}{x^2}\right)^{15}$ **2**

(c) A hot drink is placed in a closed room, where the temperature is a constant 15°C

The cooling of the drink follows the rule

$$\frac{dT}{dt} = -k(T - 15)$$

where k is a constant, t is the time in minutes and T is the temperature in °C

(i) Show that $T = 15 + Ae^{-kt}$ satisfies this equation, where A is a constant **1**

(ii) The hot drink is initially 88°C and cools to 55°C after 11 minutes. **2**

Find the value of the constants A and k, leaving in exact form

(iii) How long will it take for the drink to cool to 33°C (to the nearest second)? **2**

(d) Prove, by mathematical induction, that for integers $n \geq 1$ **3**

$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$$

Question 4 (12 marks) **[START A NEW PAGE]**

(a) NP is the normal to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$,

where N also lies on the parabola $x^2 = 4ay$

W lies on NP such that SW is parallel to the tangent to the parabola at P,

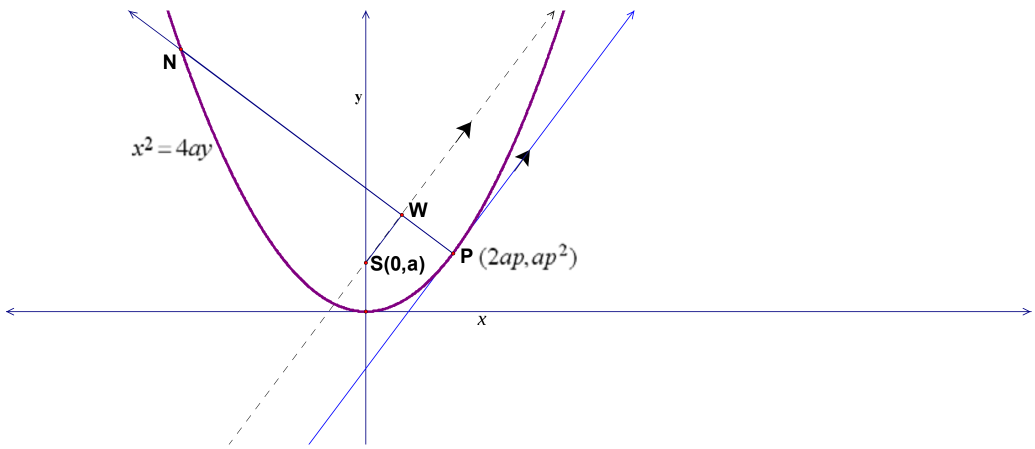
where S is the focus of the parabola $x^2 = 4ay$

You may assume the equation of the normal at P is

$x + py = ap^3 + 2ap$ (Do NOT prove this)

(i) Show that the coordinates of the point W are $(ap, ap^2 + a)$ **2**

(ii) Hence, or otherwise, find the locus of W **1**



Question 4 continues on page 6

Marks

Question 4 (continued)

(b) A particle is moving so that its displacement, x metres, from the origin is given by

$$x = 2\cos\left(3t - \frac{\pi}{6}\right)$$

where t is in seconds

(i) Show that the motion is simple harmonic **1**

(ii) Write down the period **1**

(iii) Find the velocity when the particle is first at $x = \sqrt{3}$ **2**

(c) The acceleration of a particle moving along a straight path is given by

$$\ddot{x} = -2e^{-x}$$

where x is in metres.

Initially, the particle is at the origin with a velocity of $2\,m/s$

(i) Show that $v = 2e^{-x/2}$ **2**

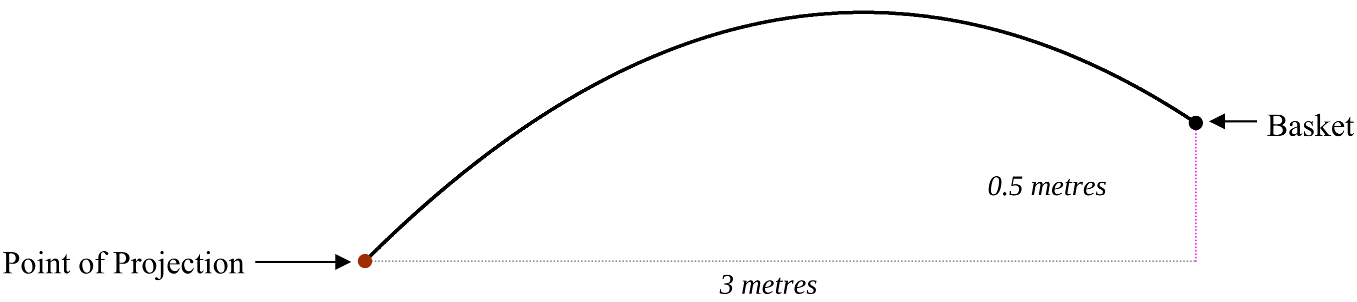
(ii) Find the equation of displacement, x, in terms of t seconds **3**

End of Question 4

Marks

Question 5 (12 marks) **[START A NEW PAGE]**

(a)



Greg is about to have a shot at goal in a game of basketball.

From the point where the ball leaves his hand, the distance to the top of the basket is 3 metres horizontally and 0.5m vertically.

Greg shoots at the optimal angle of 45°.

You may assume the equations of motion are

$x = vt\cos 45^o$ and $y = vt\sin 45^o - 5t^2$ (Do NOT prove this)

(i) Find the velocity of projection, v, required by Greg for the centre of the ball to land in the centre of the basket. **2**

(ii) Find the maximum vertical height above the basket that the ball reaches during Greg's shot. **2**

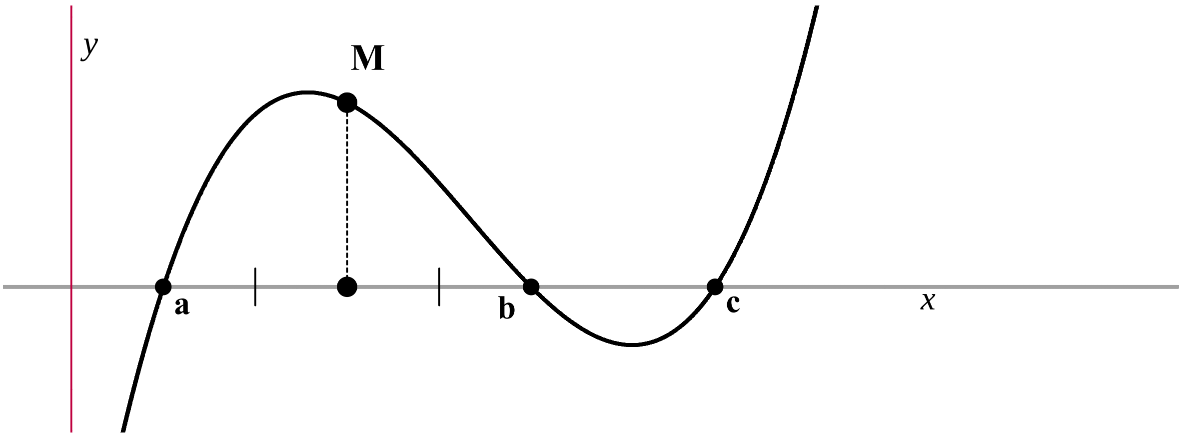
(iii) Find the speed of the ball on entry into the basket. **2**

Question 5 continues on page 6

Marks

Question 5 (continued)

(b) The graph below shows the cubic function $f(x) = (x-a)(x-b)(x-c)$ where $x = a, x = b$ and $x = c$ are the x-intercepts.



The derivative of this function is given by

$f'(x) = (x-a)(x-b) + (x-a)(x-c) + (x-b)(x-c)$ (Do NOT prove this)

which is the product rule extended, ie. $y' = uvw' + uwv' + vwu'$

The point M lies on this cubic function $y = f(x)$ such that the x-value of M

is $x = \dfrac{a+b}{2}$, ie. the mid-value of $x=a$ and $x=b$.

(i) Show that the coordinates of M are $\left(\dfrac{a+b}{2}, -\dfrac{1}{8}(a-b)^2(a+b-2c)\right)$ **2**

(ii) Show that the gradient of the tangent at M is given by **2**

$$f'\left(\frac{a+b}{2}\right) = -\frac{(a-b)^2}{4}$$

(iii) Find the gradient of the straight line passing through M and the point (c , 0). **1**

(iv) What can you conclude about the tangent at M and its x-intercept? **1**

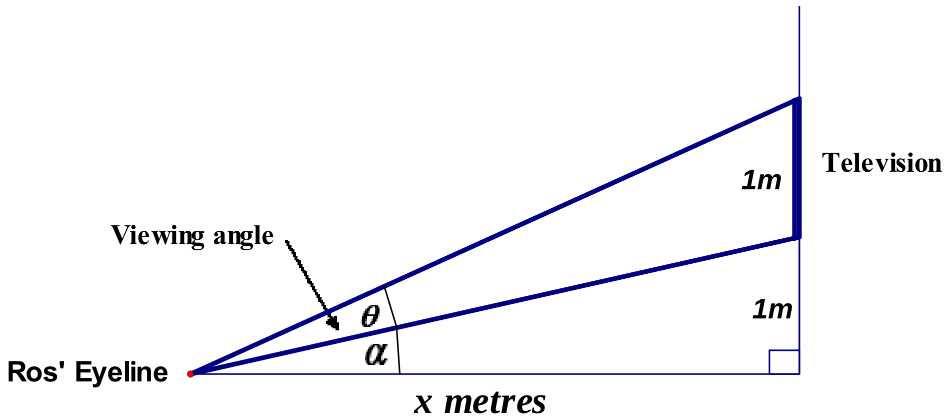
Justify your answer.

End of Question 5

Marks

Question 6 (12 marks) **[START A NEW PAGE]**

(a) Ros buys a new 1 metre tall plasma television.
She mounts it on a vertical wall, placing it so that the base of the television is 1 metre above her (horizontal) eyeline from where she sits in her favourite armchair.
Let the distance from her eye to the wall be x metres and the angle from her eye to the top and base of the television be θ (the viewing angle).
Let α be the angle of elevation to the base of the television.



(i) Show that $\theta = \tan^{-1}\left(\dfrac{2}{x}\right) - \tan^{-1}\left(\dfrac{1}{x}\right)$ **2**

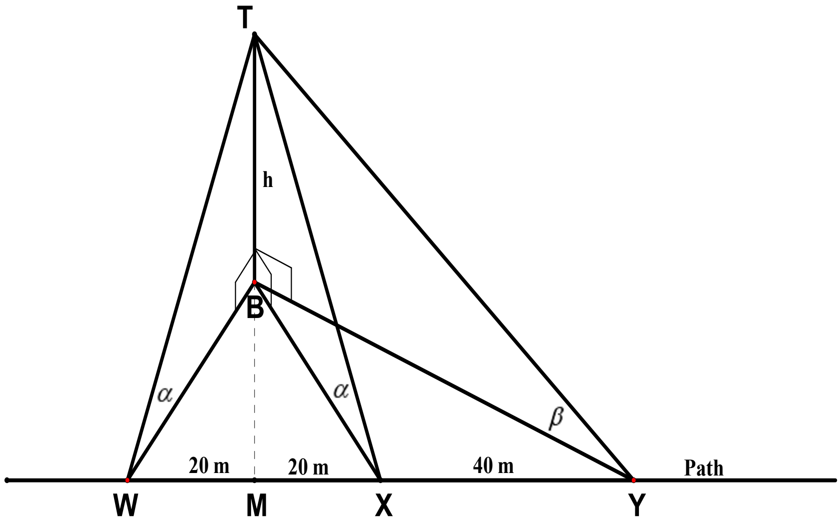
(ii) Find the value of x which gives the maximum viewing angle. **3**

(iii) Hence, find the maximum viewing angle of θ, to the nearest degree. **1**

Question 6 continues on page 10

Question 6 (continued)

(b)



Three points W, X and Y lie on a straight level path, where $WX = XY = 40\,metres$.

The base B of a flagpole TB is level with the path. M is the midpoint of WX.

The angles of elevation to the top of the flagpole from the points W, X and Y are α, α and β respectively.

(i) Prove that ΔBWX is isosceles. **2**

(ii) Find the length of BM. **1**

(iii) Hence, or otherwise, show that the height of the flagpole is given by **3**

$$h = \frac{40\sqrt{2}}{\sqrt{\cot^2\beta - \cot^2\alpha}}$$

Marks

Question 7 (12 marks) **[START A NEW PAGE]**

(a) In how many ways can five boys and two girls be arranged in a row if

(i) there are no restrictions **1**

(ii) the girls must be together **2**

(iii) there are exactly two boys separating the girls **2**

(b) Laurence was asked to sketch the graph of the curve $y = \dfrac{x}{\log_e\left(x^2\right)}$

(i) He was about to change $\log_e\left(x^2\right)$ to $2\log_e x$, but then realised this would actually alter the graph itself. Briefly explain why. **1**

(ii) Accurately describe the domain. **1**

(iii) Find the derivative of the curve. **1**

(iv) Find the two turning points of this curve and determine their nature. **2**

(v) Sketch the graph of this curve clearly labelling all critical features. **2**

End of Paper

2009 Year 12 Extension 1 Mathematics Trial HSC

Barker College.

Question 1

(a) $\left(\dfrac{kx_2+lx_1}{k+l}, \dfrac{ky_2+ly_1}{k+l}\right)$

$(-2,3)\quad(3,-4)\quad 3:-2$

$x_1\ y_1\quad x_2\ y_2\quad k\ l$

$P=\left(\dfrac{3\times3+-2\times-2}{3-2}, \dfrac{3\times-4+-2\times3}{3-2}\right)$

$=\left(\dfrac{9+4}{1}, \dfrac{-12-6}{1}\right)$

$=(13,-18)$

(b) $\lim\limits_{x\to0}\left(\dfrac{\sin5x}{5x}\times\dfrac{5}{2}\right)$

$=1\times\dfrac{5}{2}$

$=\dfrac{5}{2}$

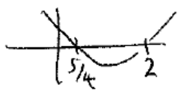
(c) $(2x-4)^2\times\dfrac{3}{(2x-4)}>-2(2x-4)^2$

$\therefore 3(2x-4)>-2(2x-4)^2$

$\therefore 2(2x-4)^2+3(2x-4)>0$

$\therefore (2x-4)[2(2x-4)+3]>0$

$\therefore (2x-4)(4x-5)>0$



$\therefore x<5/4$ or $x>2$

(d) $-\left[\sin^{-1}\left(\dfrac{x}{\sqrt2}\right)\right]_{-1}^{1}$

$=-\left[\sin^{-1}\left(\dfrac{1}{\sqrt2}\right)-\sin^{-1}\left(\dfrac{-1}{\sqrt2}\right)\right]$

$=-\left[\dfrac{\pi}{4}+\dfrac{\pi}{4}\right]$

$=-\dfrac{\pi}{2}$

(e) $\cos2x=1-2\sin^2x$

$\therefore \cos8x=1-2\sin^24x$

$\therefore \displaystyle\int_0^{\pi/12}2\sin^24x\,dx=\int_0^{\pi/12}2\times\frac{1}{2}(1-\cos8x)\,dx$

$=\displaystyle\int_0^{\pi/12}1-\cos8x\,dx$

$=\left[x-\dfrac{1}{8}\sin8x\right]_0^{\pi/12}$

$=\left(\dfrac{\pi}{12}-\dfrac{1}{8}\sin\left(\dfrac{2\pi}{3}\right)\right)-\left(0-\dfrac{1}{8}\sin0\right)$

$=\dfrac{\pi}{12}-\dfrac{1}{8}\times\dfrac{\sqrt3}{2}$

$=\dfrac{\pi}{12}-\dfrac{\sqrt3}{16}$

Question 2

(a) $u=3x^4-3$

$\therefore\dfrac{du}{dx}=12x^3\qquad\therefore\dfrac{du}{2}=6x^3\,dx$

$\therefore\displaystyle\int 6x^3\sqrt{(3x^4-3)^3}\,dx=\int\frac{1}{2}\sqrt{u^3}\,du$

$=\dfrac{1}{2}\displaystyle\int u^{3/2}\,du$

$=\dfrac{1}{2}\times\dfrac{2}{5}u^{5/2}+C$

$=\dfrac{1}{5}(3x^4-3)^{5/2}+C\quad$ or $\dfrac{\sqrt{(3x^4-3)^5}}{5}+C$

(b)(i) Domain: $-1\le\dfrac{x}{3}\le1$

$\therefore -3\le x\le3\qquad\therefore A(3,\pi)$

Range: $-\pi\le y\le\pi$

(ii) $\dfrac{dy}{dx}=2\times\sin^{-1}\left(\dfrac{x}{3}\right)+2x\times\dfrac{\frac{1}{3}}{\sqrt{1-\frac{x^2}{9}}}+2\times\dfrac{1}{2}(9-x^2)^{-\frac{1}{2}}\times-2x$

$=2\sin^{-1}\left(\dfrac{x}{3}\right)+\dfrac{2x}{3\frac{\sqrt{9-x^2}}{3}}-\dfrac{2x}{(9-x^2)^{1/2}}$

$=2\sin^{-1}\left(\dfrac{x}{3}\right)+\dfrac{2x}{\sqrt{9-x^2}}-\dfrac{2x}{\sqrt{9-x^2}}$

$=2\sin^{-1}\left(\dfrac{x}{3}\right)$

(Q2) (cont.)

(b)(iii) $\text{Area} = \int_0^3 2\sin^{-1}\left(\frac{x}{3}\right)dx$

$= \left[2x\sin^{-1}\left(\frac{x}{3}\right) + 2\sqrt{9-x^2}\right]_0^3$

$= \left[\left(2\times 3\sin^{-1}(1) + 2\sqrt{9-9}\right) - \left(2\times 0\sin^{-1}0 + 2\sqrt{9-0}\right)\right]$

$= \left[\left(6\times\frac{\pi}{2} + 0\right) - \left(0 + 2\times 3\right)\right]$

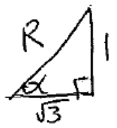
$= (3\pi - 6)$ units^2

(c)(i) $\sqrt{3}\sin t + \cos t = R\sin(t+\alpha)$

$= R\sin t\cos\alpha + R\cos t\sin\alpha$

$\therefore \sqrt{3} = R\cos\alpha \quad \therefore \cos\alpha = \frac{\sqrt{3}}{R}$

$1 = R\sin\alpha \quad \therefore \sin\alpha = \frac{1}{R}$



$\therefore R = \sqrt{3+1}$

$R = 2$

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \pi/6$

$\therefore \sqrt{3}\sin t + \cos t = 2\sin\left(t + \frac{\pi}{6}\right)$

(ii) $\sqrt{3}\sin t + \cos t = \sqrt{3}$

$\therefore 2\sin\left(t + \frac{\pi}{6}\right) = \sqrt{3}$

$\therefore \sin\left(t + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\therefore t + \frac{\pi}{6} = \frac{\pi}{3}, \pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 3\pi - \frac{\pi}{3}, \ldots$

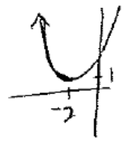
$\therefore t = \frac{\pi}{3} - \frac{\pi}{6}, \frac{2\pi}{3} - \frac{\pi}{6}, \frac{7\pi}{3} - \frac{\pi}{6}, \ldots$

$t = \frac{\pi}{6}, \frac{3\pi}{6},$ ~~$\frac{13\pi}{6}$~~

$\therefore t = \frac{\pi}{6}, \frac{\pi}{2}$

Question 3

(a)(i) $(x+2)^2 = (y-1)$ Vertex$(-2, 1)$



$x \leq -2$

(ii) Domain of $f^{-1}(x)$: $x \geq 1$

(b) General term $= {}^{15}C_r (x^4)^{15-r}\left(\frac{3}{x^2}\right)^r$

$= {}^{15}C_r\, x^{60-4r}\, 3^r (x^{-2})^r$

$= {}^{15}C_r\, 3^r x^{60-6r}$

constant term when $60 - 6r = 0$

$\therefore r = 10$

$\therefore$ constant term is ${}^{15}C_{10}\, 3^{10}$

(c)(i) $T - 15 = Ae^{-kt}$

$\frac{dT}{dt} = 0 + -kAe^{-kt}$

$= -k(T-15)$

(ii) When $t=0$, $T = 88$

$\therefore 88 = 15 + Ae^0$

$\therefore A = 88 - 15$

$= 73$

when $t = 11$, $T = 55$

$\therefore 55 = 15 + 73e^{-11k}$

$\therefore 40 = 73e^{-11k}$

$\therefore \frac{40}{73} = e^{-11k}$

$\therefore -11k = \log_e\left(\frac{40}{73}\right)$

$\therefore k = -\frac{1}{11}\ln\left(\frac{40}{73}\right)$

(iii) $33 = 15 + 73e^{-kt}$

$\therefore 18 = 73e^{-kt}$

$\therefore \frac{18}{73} = e^{-kt}$

$\therefore -kt = \ln\left(\frac{18}{73}\right)$

$\therefore t = \frac{\ln\left(\frac{18}{73}\right)}{+\frac{1}{11}\ln\left(\frac{40}{73}\right)} = 25{\cdot}6\text{ min} \doteqdot 25\text{ min } 36\text{ sec}$

(Q3) (cont.)

(d) Prove true for $n=1$

$LHS = (2-1)^2 = 1$ $\quad RHS = \frac{1}{3}\times 1\times(2-1)\times(2+1) = 1$

$\therefore$ True for $n=1$

Assume true for $n=k$

ie. $1^2+3^2+\dots+(2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$

Now prove true for $n=k+1$

$RHS = \frac{1}{3}k(2k-1)(2k+1) + (2(k+1)-1)^2$

$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$

$= \frac{1}{3}(2k+1)[k(2k-1)+3(2k+1)]$

$= \frac{1}{3}(2k+1)(2k^2+5k+3)$

$= \frac{1}{3}(2k+1)(2k+3)(k+1)$

$= \frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)$

$\therefore$ Statement is true for $n=k+1$

Since statement is true for $n=1$ and $n=k+1$, then it is true for $n=2,3,4,\dots$ ie. for all integers $n\geq 1$

Question 4

(a)(i) Gradient of SW $= p$, y-int $= a$

$\therefore$ Eqn of SW is $y=px+a$

$x+py = ap^3+2ap$
$y = px+a$

$\therefore x+p(px+a) = ap^3+2ap$

$\therefore x+p^2x+ap = ap^3+2ap$

$\therefore p^2x+x = ap^3+ap$

$\therefore x(p^2+1) = ap(p^2+1)$

$\therefore x = ap$

$y = p\times ap + a$
$= ap^2+a$ $\quad \therefore W(ap, ap^2+a)$

(ii) $ap = x$ $\quad \therefore p = \frac{x}{a}$

$y = ap^2+a$

$\therefore y = a\left(\frac{x}{a}\right)^2 + a$

$y = \frac{x^2}{a} + a$ $\quad$ (OR $x^2 = a(y-a)$)

(b)(i) $x = 2\cos\left(3t-\frac{\pi}{6}\right)$

$\dot{x} = 2\times -3\sin\left(3t-\frac{\pi}{6}\right)$

$\ddot{x} = -2\times 9\cos\left(3t-\frac{\pi}{6}\right)$

$= -9\times 2\cos\left(3t-\frac{\pi}{6}\right)$

$\ddot{x} = -9x$ which is in the form $\ddot{x} = -n^2x$ where $n=3$

$\therefore$ SHM

(ii) Period $= \frac{2\pi}{3}$

(iii) $\sqrt{3} = 2\cos\left(3t-\frac{\pi}{6}\right)$

$\cos\left(3t-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\therefore 3t-\frac{\pi}{6} = \frac{\pi}{6}, \dots$

$\therefore 3t = \frac{\pi}{3}, \dots$

$\therefore t = \frac{\pi}{9}, \dots$

when $t=\frac{\pi}{9}$, $v = -6\sin\left(3\times\frac{\pi}{9}-\frac{\pi}{6}\right)$

$= -6\sin\left(\frac{\pi}{6}\right)$

$= -6\times\frac{1}{2}$

$= -3$ m/s

(OR $3t-\frac{\pi}{6} = -\frac{\pi}{6}$
$3t = 0$
$\therefore t = 0$
$\therefore v = -6\sin\left(-\frac{\pi}{6}\right)$
$= -6\times -\frac{1}{2}$
$= 3$ m/s)

(c)(i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2e^{-x}$

$\therefore \frac{1}{2}v^2 = 2e^{-x}+c$

When $x=0$, $v=2$

$\therefore \frac{4}{2} = 2e^0+c$

$\therefore 2 = 2+c$ $\quad \therefore c=0$

$\therefore \frac{1}{2}v^2 = 2e^{-x}$

$\therefore v^2 = 4e^{-x}$

$\therefore v = \pm\sqrt{4e^{-x}}$

But when $x=0$, $v=2$

$\therefore v = +2\sqrt{e^{-x}}$

$\therefore v = 2(e^{-x})^{1/2}$ $\quad \therefore v = 2e^{-\frac{x}{2}}$

Q4 (cont.)

(c)(ii) $\dfrac{dx}{dt} = 2e^{-\frac{x}{2}}$

$\therefore \dfrac{dt}{dx} = \dfrac{1}{2e^{-x/2}}$

$= \frac{1}{2}e^{x/2}$

$\therefore t = \frac{1}{2} \times 2e^{\frac{x}{2}} + c$

$t = e^{x/2} + c$

When $t=0$, $x=0$

$0 = e^{0} + c \quad \therefore c = -1$

$\therefore t = e^{x/2} - 1$

$\therefore t+1 = e^{\frac{x}{2}}$

$\therefore \frac{x}{2} = \log_e(t+1)$

$\therefore x = 2\ln(t+1)$

Question 5

(a)(i) $x = \dfrac{vt}{\sqrt{2}}$ and $y = \dfrac{vt}{\sqrt{2}} - 5t^2$

$x = 3$ gives $3 = \dfrac{vt}{\sqrt{2}}$

$\therefore t = \dfrac{3\sqrt{2}}{v}$

And $y = 0.5$

$\therefore 0.5 = \dfrac{v}{\sqrt{2}} \times \dfrac{3\sqrt{2}}{v} - 5 \times \left(\dfrac{3\sqrt{2}}{v}\right)^2$

$0.5 = 3 - 5 \times \dfrac{9 \times 2}{v^2}$

$\therefore -2.5 = -\dfrac{90}{v^2}$

$\therefore 2.5v^2 = 90$

$\therefore v^2 = 36$

$\therefore v = 6$ m/s (since $v > 0$)

(ii) Max height when $\dot{y} = 0$

$y = \dfrac{6t}{\sqrt{2}} - 5t^2$

$\therefore \dot{y} = \dfrac{6}{\sqrt{2}} - 10t$

$0 = \dfrac{6}{\sqrt{2}} - 10t$

$10t = \dfrac{6}{\sqrt{2}}$

$t = \dfrac{3}{5\sqrt{2}}$

(a)(ii)(cont.) When $t = \dfrac{3}{5\sqrt{2}}$

$y = \dfrac{6}{\sqrt{2}} \times \dfrac{3}{5\sqrt{2}} - 5 \times \left(\dfrac{3}{5\sqrt{2}}\right)^2$

$= \dfrac{18}{5 \times 2} - 5 \times \dfrac{9}{25 \times 2}$

$= 1.8 - 0.9$

$= 0.9$ m

ie. 0.4m (or 40cm) above basket

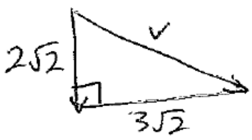
(iii) $x = 3$ in basket

$\therefore 3 = \dfrac{6t}{\sqrt{2}} \quad \therefore t = \dfrac{3\sqrt{2}}{6} = \dfrac{\sqrt{2}}{2}$ sec

So $\dot{x} = \dfrac{6}{\sqrt{2}}$ and $\dot{y} = \dfrac{6}{\sqrt{2}} - \dfrac{10 \times \sqrt{2}}{2}$

$\dot{x} = \dfrac{6\sqrt{2}}{2} = 3\sqrt{2}$

$\dot{y} = \dfrac{6\sqrt{2}}{2} - 5\sqrt{2} = -2\sqrt{2}$



$\therefore v^2 = (2\sqrt{2})^2 + (3\sqrt{2})^2$

$v^2 = 4 \times 2 + 9 \times 2$

$v^2 = 26$

$\therefore$ Speed $= \sqrt{26}$ m/s

(b)(i) $f\left(\dfrac{a+b}{2}\right) = \left(\dfrac{a+b}{2} - a\right)\left(\dfrac{a+b}{2} - b\right)\left(\dfrac{a+b}{2} - c\right)$

$= \left(\dfrac{a+b-2a}{2}\right)\left(\dfrac{a+b-2b}{2}\right)\left(\dfrac{a+b-2c}{2}\right)$

$= \frac{1}{2}(b-a)\frac{1}{2}(a-b)\frac{1}{2}(a+b-2c)$

$= \frac{1}{8}(b-a)(a-b)(a+b-2c)$

$= -\frac{1}{8}(a-b)(a-b)(a+b-2c)$

$= -\frac{1}{8}(a-b)^2(a+b-2c)$

(ii) $f'\left(\dfrac{a+b}{2}\right) = \left(\dfrac{a+b}{2} - a\right)\left(\dfrac{a+b}{2} - b\right) + \left(\dfrac{a+b}{2} - a\right)\left(\dfrac{a+b}{2} - c\right) + \left(\dfrac{a+b}{2} - b\right)\left(\dfrac{a+b}{2} - c\right)$

$\therefore f'\left(\frac{a+b}{2}\right) = \left(\dfrac{a+b-2a}{2}\right)\left(\dfrac{a+b-2b}{2}\right) + \left(\dfrac{a+b-2a}{2}\right)\left(\dfrac{a+b-2c}{2}\right) + \left(\dfrac{a+b-2b}{2}\right)\left(\dfrac{a+b-2c}{2}\right)$

$= \frac{1}{4}(b-a)(a-b) + \frac{1}{4}(b-a)(a+b-2c) + \frac{1}{4}(a-b)(a+b-2c)$

$= -\frac{1}{4}(a-b)(a-b) - \frac{1}{4}(a-b)(a+b-2c) + \frac{1}{4}(a-b)(a+b-2c)$

$= -\frac{1}{4}(a-b)^2$

Q5 (cont.)

(b)(iii) $m = \dfrac{\left(-\frac{1}{8}(a-b)^2(a+b-2c)-0\right)}{\left(\frac{a+b}{2}-c\right)} \times \dfrac{2}{2}$

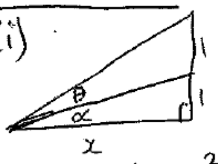
$= \dfrac{-\frac{1}{4}(a-b)^2(a+b-2c)}{(a+b-2c)}$

$= -\frac{1}{4}(a-b)^2$

(iv) Since the gradients in Parts (ii) & (iii) are equal, the tangent at M must pass through $x = c$

Question 6

(a)(i)



$\tan\alpha = \frac{1}{x}$

$\therefore \alpha = \tan^{-1}\left(\frac{1}{x}\right)$

$\tan(\theta+\alpha) = \frac{2}{x}$

$\therefore \theta + \alpha = \tan^{-1}\left(\frac{2}{x}\right)$

$\therefore \theta = \tan^{-1}\left(\frac{2}{x}\right) - \alpha$

$\therefore \theta = \tan^{-1}\left(\frac{2}{x}\right) - \tan^{-1}\left(\frac{1}{x}\right)$

(ii) $\theta = \tan^{-1}(2x^{-1}) - \tan^{-1}(x^{-1})$

$\therefore \dfrac{d\theta}{dx} = \dfrac{-2x^{-2}}{1+\frac{4}{x^2}} - \dfrac{-x^{-2}}{1+\frac{1}{x^2}}$

$= \dfrac{\frac{-2}{x^2}}{1+\frac{4}{x^2}} + \dfrac{\frac{1}{x^2}}{1+\frac{1}{x^2}}$

$= \dfrac{-2}{x^2+4} + \dfrac{1}{x^2+1}$

Maximum when $\frac{d\theta}{dx} = 0$

$\therefore 0 = \dfrac{-2}{x^2+4} + \dfrac{1}{x^2+1}$

$\therefore \dfrac{2}{x^2+4} = \dfrac{1}{x^2+1}$

$\therefore 2x^2+2 = x^2+4$

$\therefore x^2 = 2$

$\therefore x = \sqrt{2}$ or $-\sqrt{2}$

Test $x = \sqrt{2}$ (since $x > 0$)

x	1	$\sqrt{2}$	2
$\frac{d\theta}{dx}$	$\frac{1}{10}$	0	$-\frac{1}{20}$
	/	—	\

$\therefore$ Max when $x = \sqrt{2}$

(iii) Max viewing angle $= \tan^{-1}\left(\frac{2}{\sqrt{2}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= \tan^{-1}(\sqrt{2}) - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= 54°44'8.2'' - 35°15'51.8''$

$= 19°28'16.39''$

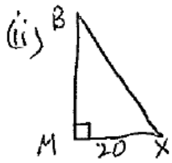
$\doteqdot 19°$ (nearest degree)

(b)(i) $\tan\alpha = \frac{h}{WB}$ $\therefore WB = \frac{h}{\tan\alpha} = h\cot\alpha$

Also, $\tan\alpha = \frac{h}{BX}$ $\therefore BX = \frac{h}{\tan\alpha} = h\cot\alpha$

$\therefore \triangle BWX$ is isosceles ($WB = BX$ i.e. 2 equal sides)

(ii)

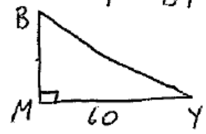


$BM = \sqrt{h^2\cot^2\alpha - 20^2}$

$= \sqrt{h^2\cot^2\alpha - 400}$

(OR $BM = \sqrt{h^2\cot^2\beta - 3600}$)

(iii) $\tan\beta = \frac{h}{BY}$ $\therefore BY = h\cot\beta$



$BY^2 = BM^2 + MY^2$

$MY = 20 + 40$
$= 60$

$h^2\cot^2\beta = (h^2\cot^2\alpha - 400) + 60^2$

$h^2\cot^2\beta = h^2\cot^2\alpha - 400 + 3600$

$h^2\cot^2\beta - h^2\cot^2\alpha = 3200$

$h^2(\cot^2\beta - \cot^2\alpha) = 3200$

$h^2 = \dfrac{3200}{(\cot^2\beta - \cot^2\alpha)}$

$\therefore h = \dfrac{\sqrt{3200}}{\sqrt{\cot^2\beta - \cot^2\alpha}}$ (since $h > 0$)

$\therefore h = \dfrac{40\sqrt{2}}{\sqrt{\cot^2\beta - \cot^2\alpha}}$

Question 7

(a)(i) $7! = {}^7P_7 = 5040$ ways

(ii) GGBBBBB, BGGBBBB, BBGGBBB, BBBGGBB, BBBBGGB, BBBBBGG

Each of these can be arranged $5! \times 2!$ ways

$\therefore$ Total $= 6! \times 2!$

$= 1440$ ways

(iii) GBBGBBB, BGBBGBB, BBGBBGB, BBBGBBG

Each of these can be arranged $5! \times 2!$ ways

$\therefore$ Total $= 4 \times 5! \times 2!$

$= 960$ ways

(b)(i) $\log_e(x^2)$ has a domain of all real x but $x \neq 0$

$2\log_e x$ only has a domain of $x > 0$, so half of the curve would be deleted/lost

(ii) $x \neq 0$ since $\log_e 0$ is undefined

And since $\log_e 1 = 0$ and $\frac{x}{0}$ is undefined thus $x \neq \pm 1$

$\therefore$ Domain is all real x but $x \neq 0, x \neq \pm 1$

(iii) $\frac{dy}{dx} = \frac{(\ln x^2) \times 1 - x \times \frac{2x}{x^2}}{(\ln x^2)^2}$

$= \frac{\ln x^2 - 2}{(\ln x^2)^2}$

(iv) stat pts when $\frac{dy}{dx} = 0$

$\therefore 0 = \frac{\ln x^2 - 2}{(\ln x^2)^2}$

$\therefore 2 = \ln x^2$

$\therefore x^2 = e^2$

$\therefore x = \pm\sqrt{e^2}$

$\therefore x = e$ or $x = -e$

$y = \frac{e}{2}$ $\quad y = \frac{-e}{2}$

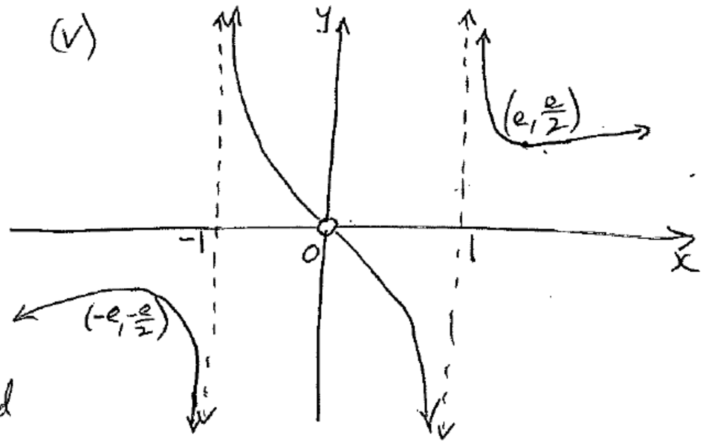
x	-3	$-e$	-2
y'	$\frac{\ln 9 - 2}{(\ln 9)^2} \doteqdot 0.04$	0	$\frac{\ln 4 - 2}{(\ln 4)^2} \doteqdot -0.32$
	/	—	\

$\therefore$ Max turning pt @ $\left(-e, -\frac{e}{2}\right)$

x	2	e	3
y'	$\frac{\ln 4 - 2}{(\ln 4)^2} \doteqdot -0.32$	0	$\frac{\ln 9 - 2}{(\ln 9)^2} \doteqdot 0.04$
	\	—	/

$\therefore$ Min turning pt @ $\left(e, \frac{e}{2}\right)$

(v)



When $x = \frac{1}{2}$, $y = \frac{0.5}{\ln 0.25} < 0$

When $x = -\frac{1}{2}$, $y = \frac{-0.5}{\ln 0.25} > 0$